Differential Equations: One Course in a Lifetime of Modeling

  

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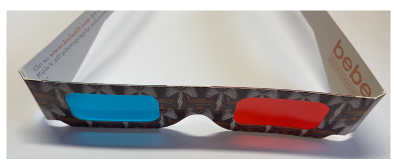
Modeling is essential for understanding and bettering our lives
EXPLANATORY vs DESCRIPTIVE -- GOALS and VALUES

Slide 1



Red-cyan 3D (anaglyph) eyeglasses

The web site https://justaddmath.org/simiode-expo-2023/ supplements this discussion. You can reach it by clicking the link or using the QR code on slide 1 above with your mobile device. This session is intended for active participation. If you have a pair, or better yet two pairs, of Polaroid™ (polarizing) sunglasses handy then it would be great to get them before the talk begins. A pair of 3D red-cyan (anaglyph) eyeglasses would also be great. A clear water glass filled halfway with water would be good, as would a washbasin partially filled with water. See slide 1 above. If you have the web site open you can get a preview of the everyday experiments we and our students can do with things found around the house.

The slide above captures the essence of this session. We want to empower our students to use modeling to help understand our world and change it for the better. Ordinary and partial differential equations are just two courses in a lifetime of modeling in math classes and across the curriculum – modeling that goes beyond description to explanation and understanding and that recognizes the importance of goals and values for decision-making.

Three Points

- Ordinary and Partial Differential Equations are one paradigm in a growing modeling toolkit or repertoire. Use the whole toolkit. Qualitative methods.
- The primary purpose of modeling or the scientific method is building understanding and giving us the power to change our world
- Modeling is lifetime, everyday, everywhere (around the house below)

The Mid-Hudson Bridge

Joseph Bertolozzi

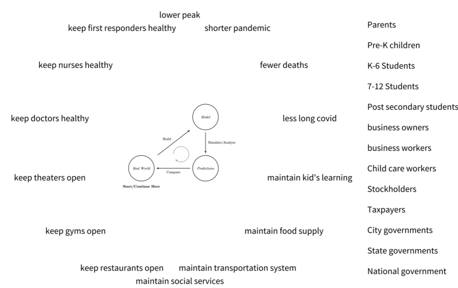
Slide 2

The three points above are the key take-aways from this session. The third one is particularly important – we want our students to use modeling throughout their lives, everyday and everywhere. I live near the Mid Hudson Bridge. Bridges like this are great places to see modeling in action – from the curve of the suspension cables to the role of links in networks. Last fall I was walking across the bridge and happened to meet the composer Joseph Bertolozzi as he was using drumsticks to play music with the bridge as his set of drums. I got a great mini-course in PDEs in action. You can learn more at https://josephbertolozzi.com. You can learn a lot about PDEs with a pair of drumsticks and things found around your house or classroom. Of course, you don't even need real drumsticks.

Open Discussion

Today (this room) after five minute break

- Experiments and observations around the house.
- Experiences with ChatGPT, text-based image search and generation.
- Ideas for Modeling for High Stakes Decisions.



Slide 3

After the first part of this session we'll break for five minutes and then return for an open discussion about the three points on this slide. The diagram at the bottom of the slide will be central to our discussion of the third point. This is just a "teaser." We'll talk more about it (a larger, easier to read version) later.

If you're reading these notes you should think about these three points and maybe even discuss them with friends.

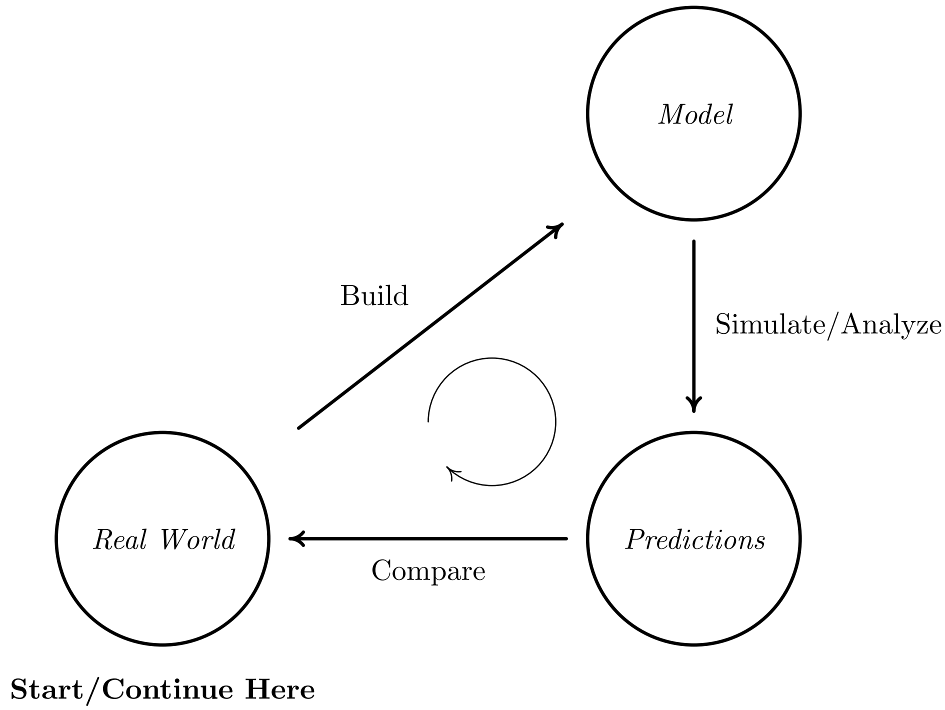


Figure 1: The modeling cycle

Slide 4

This is the single most important slide today. Modeling is fundamentally an iterative process. We begin in the real world and build a model to help us understand that world. Then we analyze, often using simulations, that model to see its predictions or implications, Then we compare these predictions with what we observe in the real world. Usually we then build another model and repeat the cycle. We start with very simple models to get some traction on the problem and then build better and more useful models. Most math courses focus on the second step – analysis – but as modelers the first and third steps are equally, perhaps more, important. The key to modeling is the dance between the real world and our models.

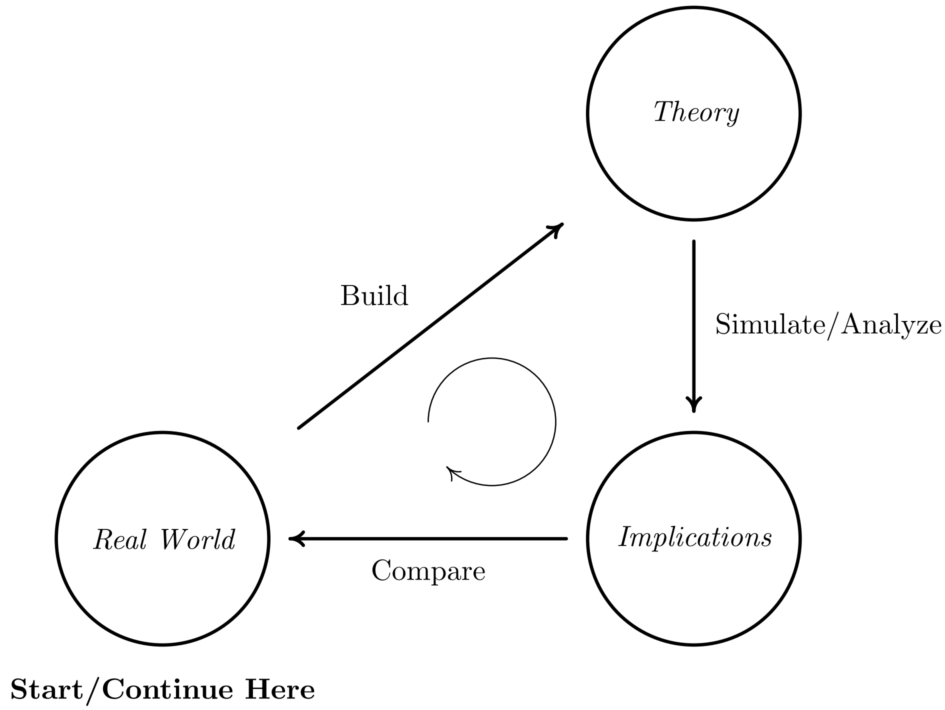
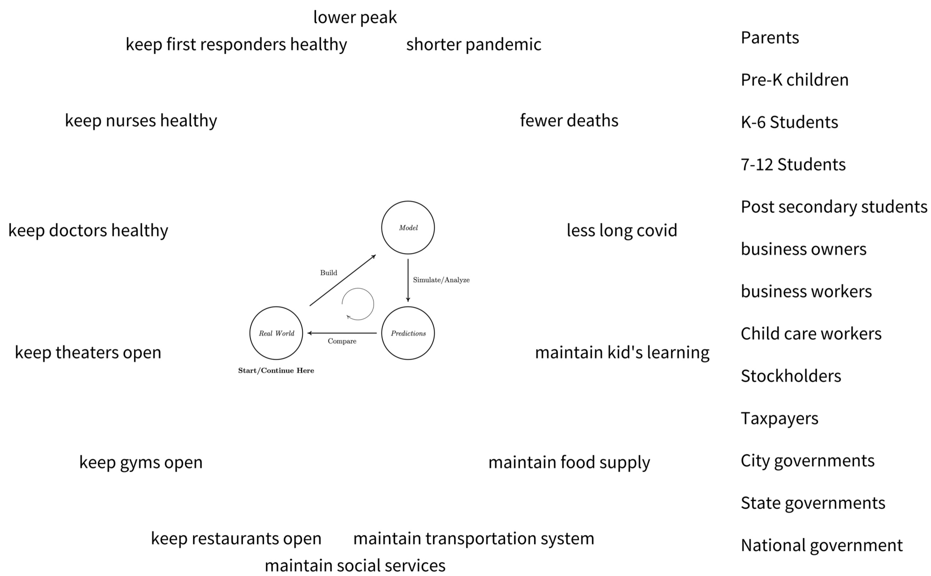


Figure 2: The Scientific Method

Slide 5

We could just as well call this the scientific method.



Modeling, Goals and Stakeholders (e.g. COVID-19)

Slide 6

We'll use the COVID-19 pandemic as an example today. Science and modeling play a central role in informing our personal and public policy decisions as we battle the pandemic. Indeed, many public figures say that we should "follow the science" but this is really misleading. As with most significant personal and public policy decisions, the decisions we make regarding the COVID-19 pandemic have many different, and sometimes possibly conflicting, goals – for example, slowing the spread of COVID-19 and maintaining important activities like children's learning. The most difficult decisions can be apparently zero-sum decisions and it is incredibly important to find options that break the zero-sum trap. Decision-making is complicated by many different and often very noisy stakeholders, each with their own set of goals and priorities. The figure on this slide with the modeling cycle surrounded by goals and flanked by stakeholders enables us to model real-world decision-making.

This framework can also help us maintain thoughtful discussions in the current highly polarized environment when far too many public figures construct alternate facts that mask real differences in the values that determine how we prioritize different goals. During the discussion session I hope we can exchange ideas about how this framework can work in classrooms. What successes and problems have you had working with high-stakes, controversial classroom discussions?

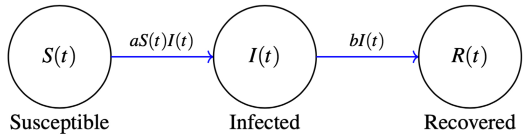


Figure 7.1: Susceptible, recovered, and infected (SIR) compartmental model of an epidemic.

As in any compartment model, we must specify the rate at which the quantities of interest (in this case, people) move between the compartments. In the classic SIR model, susceptible and infected people interact at a rate that is proportional to the product SI. The reasoning is that, for any fixed value of S, if the value of I is doubled then the number of interactions between susceptible and infected people should double, and a similar result should hold if I is fixed but S is doubled; the quantity SI captures this observation. Moreover, we suppose in the model that each such interaction carries a fixed risk of the susceptible person becoming infected and moving from the S to the I compartment. This is captured by the $aS(t)I(t)$ label above the arrow from the S compartment to the I compartment in Figure 7.1: the likelihood of infection is proportional to the number of interactions between the susceptible (S) and the infected (I) populations. The constant of proportionality a depends, for example, on the infectiousness of the disease. Movement from the I compartment to the R compartment is assumed to occur at a rate proportional to the number of infected people: all else being equal, if there are twice as many infected people then the number of people getting better per unit time should double.

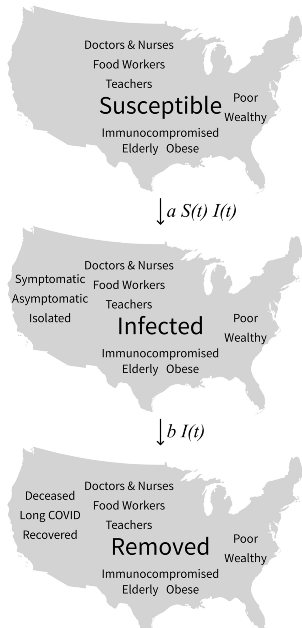
With these observations we can posit the model

$$\begin{aligned}\dot{S} &= -aSI \\ \dot{I} &= aSI - bI \\ \dot{R} &= bI.\end{aligned} \tag{7.6}$$

Differential Equations: A Toolbox for Modeling the World
by Kurt Bryan

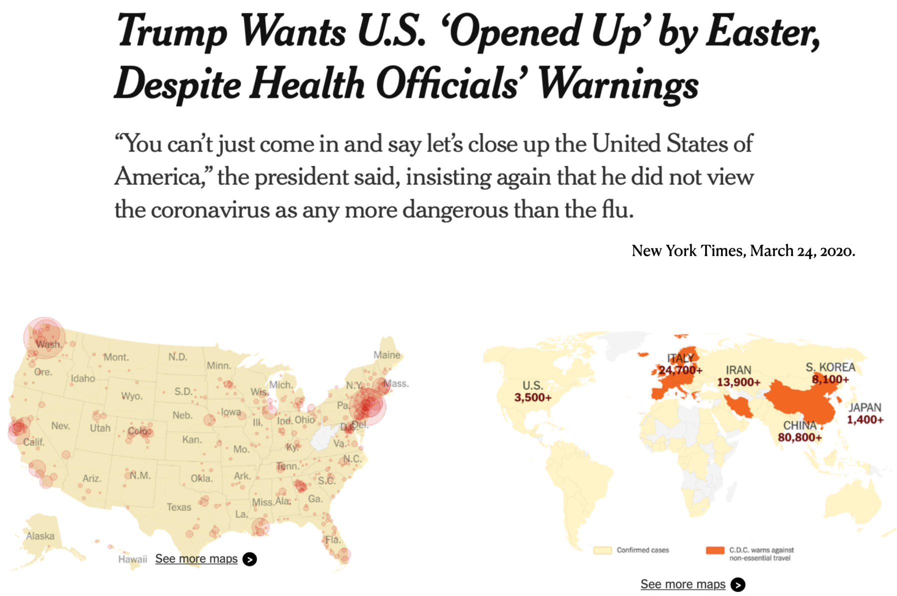
Slide 7

This is a typical and very nice introduction from Kurt Bryan's excellent SIMIODE book, *Differential Equations: A Toolbox for Modeling the World* to the SIR model that is the first iteration for much of the pandemic modeling. I'd like to suggest replacing the figure at the top by the figure on the next slide.



Slide 8

Notice that this figure emphasizes the way that the SIR model is a crude first step. It has all the information we need for the first model and at the same time suggests how crude that model is and some of the ways it might be improved.



Slide 9

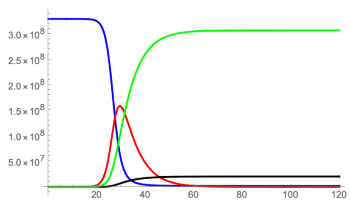
I'm just putting this slide up to remind us all of where we were in early 2020, when many of us pivoted to including modeling the pandemic, often in courses that had gone online.

$$
\begin{aligned}
S' &= -\alpha S\left(\frac{I}{S+I+R}\right)\\
I' &= \alpha S\left(\frac{I}{S+I+R}\right) - \beta I - \gamma I\\
R' &= \beta I\\
D' &= \gamma I
\end{aligned}
$$

Slide 10

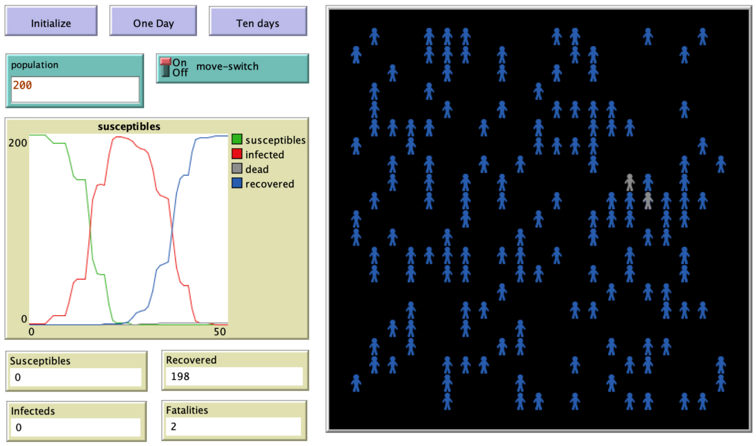
This is a simple second iteration, adding a second compartment or variable, D, for deceased and using R for recovered, rather than removed.

```
alpha = 0.80;
beta = 0.15;
gamma = 0.01;
endTime = 120;
Clear[susceptible, infected, recovered, deceased]
solution =
  NDSolve[{susceptible'[t] == - alpha * susceptible[t] * (infected[t]/(infected[t] + susceptible[t] + recovered[t])),
       infected'[t] == alpha * susceptible[t] * (infected[t]/(infected[t] + susceptible[t] + recovered[t])) - (beta + gamma) * infected[t],
       recovered'[t] == beta * infected[t],
       deceased'[t] == gamma * infected[t],
       susceptible[0] == 330000000,
       infected[0] == 10,
       deceased[0] == 0,
       recovered[0] == 0}, {susceptible[t], infected[t], recovered[t], deceased[t]}, {t, 0, endTime}];
susceptible[t_] = susceptible[t] /. solution[[1]];
infected[t_] = infected[t] /. solution[[1]];
recovered[t_] = recovered[t] /. solution[[1]];
deceased[t_] = deceased[t] /. solution[[1]];
Plot[{susceptible[t], infected[t], deceased[t], recovered[t]}, {t, 0, endTime},
 PlotStyle -> {{Blue, Thick}, {Red, Thick}, {Black, Thick}, {Green, Thick}}]
deceased[endTime]
```



Slide 11

This slide shows a *Mathematica* notebook emphasizing both the mathematical power and the expressive power of numerical methods.

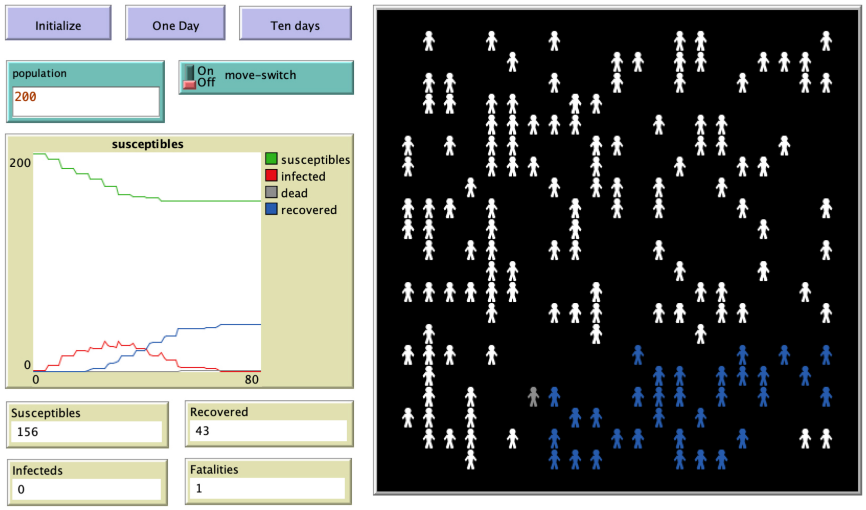


Agent-Based Modeling
Object-Oriented Programming

NetLogo

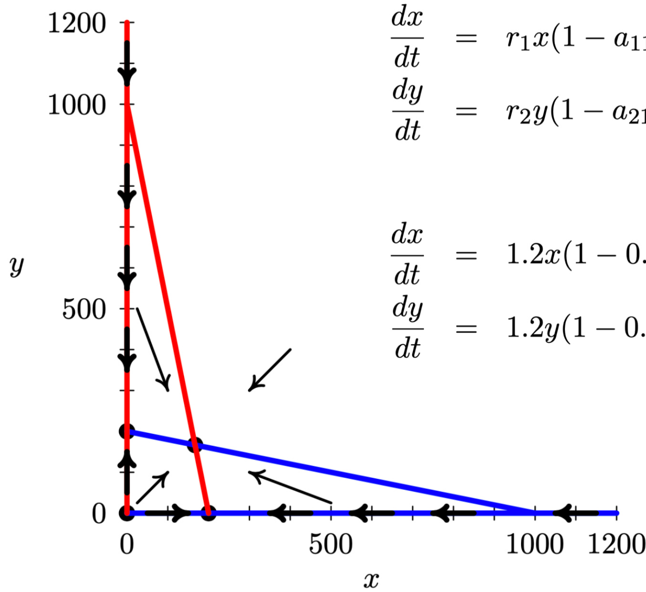
Slide 12

But, differential equations are just one tool in our modeling toolbox. Other tools include discrete dynamical systems and agent-based modeling, which is often supported by object-oriented programming. This screenshot shows one exercise using NetLogo from my own spring 2020 course. Together with the screenshot shown on the next page my students could explore possible impacts of shutting down certain activities. Note the "move-switch." In this first run it is on and agents move from their residential area to a place of work once each day. Over the course of one run there are two fatalities and 198 agents who were infected and then recovered.



Slide 13

Shutting down some activities can slow the spread of a pandemic. This is a first, crude, step in building models that can help us make important decisions. In this second run, the “move-switch” is off and agents work from home. Over the course of one run there is only one fatality and 43 agents who were infected and then recovered. This is just a “toy” model but illustrates the principle behind working and studying from home.



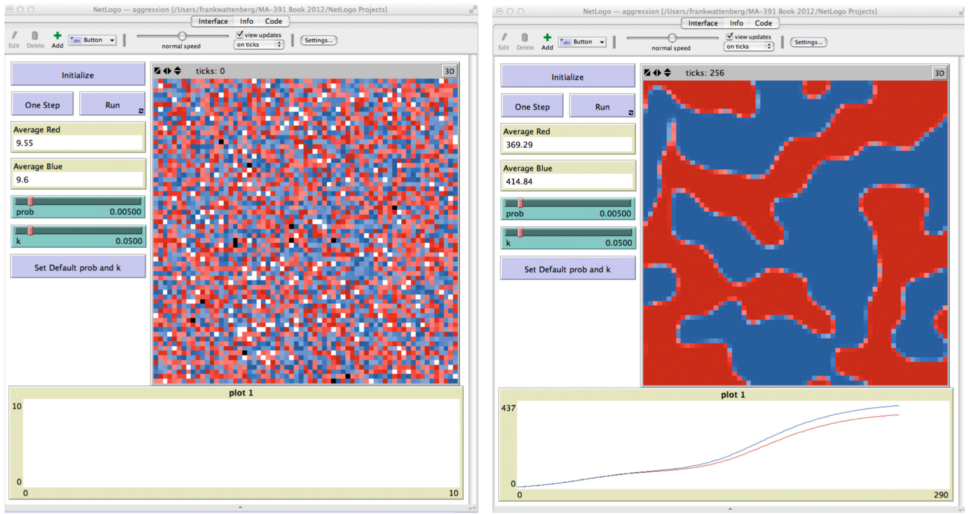
$$\frac{dx}{dt} = r_1 x(1 - a_{11}x - a_{12}y)$$
$$\frac{dy}{dt} = r_2 y(1 - a_{21}x - a_{22}y)$$

$$\frac{dx}{dt} = 1.2x(1 - 0.005x - 0.001y)$$
$$\frac{dy}{dt} = 1.2y(1 - 0.001x - 0.005y)$$

Slide 14

This is a typical Lotka-Volterra model for two species competing for some but not all of the resources in a shared habitat. Notice that the two intraspecies coefficients (a_{11} and a_{22}) have smaller magnitude than the two interspecies coefficients (a_{12} and a_{21}) and that there is an attracting equilibrium where the two species share the habitat.

On the next slide we look at a model based on a Lotka-Volterra Model of aggressive competition – the interspecies coefficients have higher magnitude than the intraspecies coefficients. In this Lotka-Volterra model the equilibrium where both species coexist is repelling and one species usually wipes out the other one.



$$\frac{dx}{dt} = 1.2x\,(1 - 0.001x - 0.005y)$$

$$\frac{dy}{dt} = 1.2y\,(1 - 0.005x - 0.001y)$$

Slide 15

But, again using NetLogo and an agent-based model in which individuals interact only with their neighbors we see that the two species can coexist with each in its own territory. As this model evolves the boundaries between territories approach straight lines.

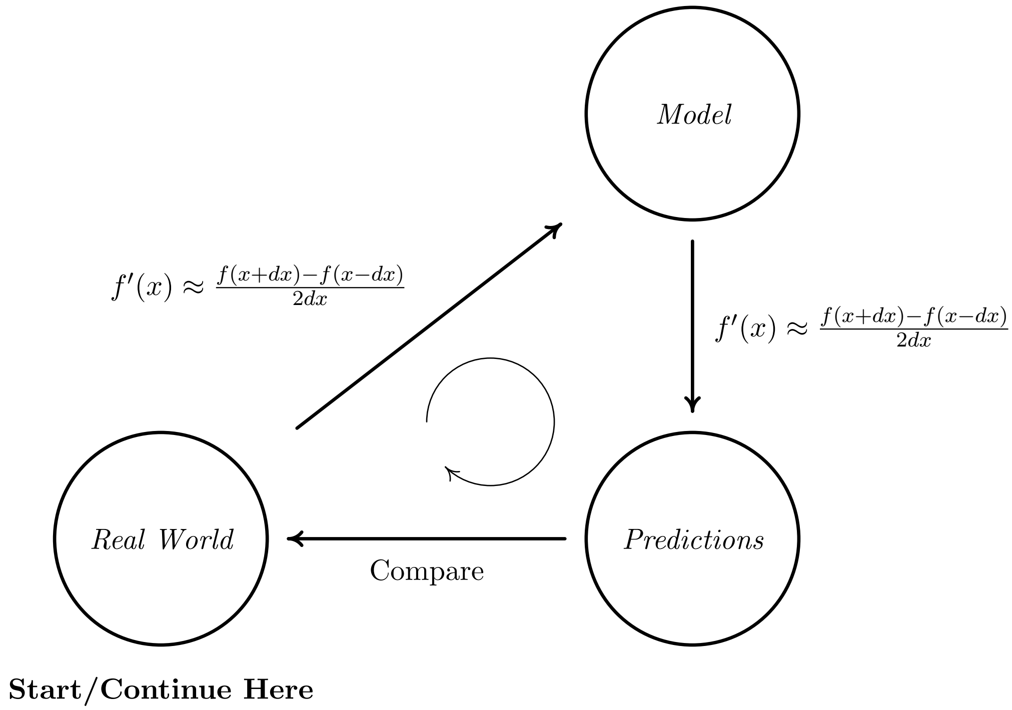


Figure 3: Dual Use

Slide 16

These are not competing tools – they are synergistic – and the synergy enters the modeling cycle in two ways. The equations like the one below

$$f'(x) \approx \frac{f(x+dx)-f(x-dx)}{2dx},$$

which is one finite difference approximation of $f'(x)$ are used both for analysis using numerical solutions of differential equations and for using discrete models to build continuous ones.

Quantum phenomena and quantum computing will challenge our modeling ability and provide new modeling paradigms.

We need to understand these. Where is Isaac Asimov when we need him?

Superposition and Quantum Entanglement.

Slide 17

Physical models often provide useful paradigms for modeling in the social sciences. For example, we talk about the "inertia" of an organization or the "momentum" of a political campaign. Quantum phenomena will provide new and, I believe, very useful paradigms. For example, the reigning paradigm for the red-blue divide in our country is rooted firmly in the traditional "bit" – either zero or one. But, I believe this paradigm is fundamentally wrong and socially destructive. Most of us are more like "qubits" then "bits." We are complicated higher dimensional individuals who may become either red or blue depending on how we are forced to choose in a zero-one one dimensional situation.

Quantum phenomena, like superposition and quantum entanglement, are fundamentally difficult to understand in large part because they are far from and even contradict our everyday experience but also because the models we have rely heavily on formal mathematics. We desperately need Isaac Asimov. His book *The Collapsing Universe* is one of my favorite science books. Modeling – that is, building understanding of – quantum phenomena is both imperative and difficult.

An Example of the Scientific Method

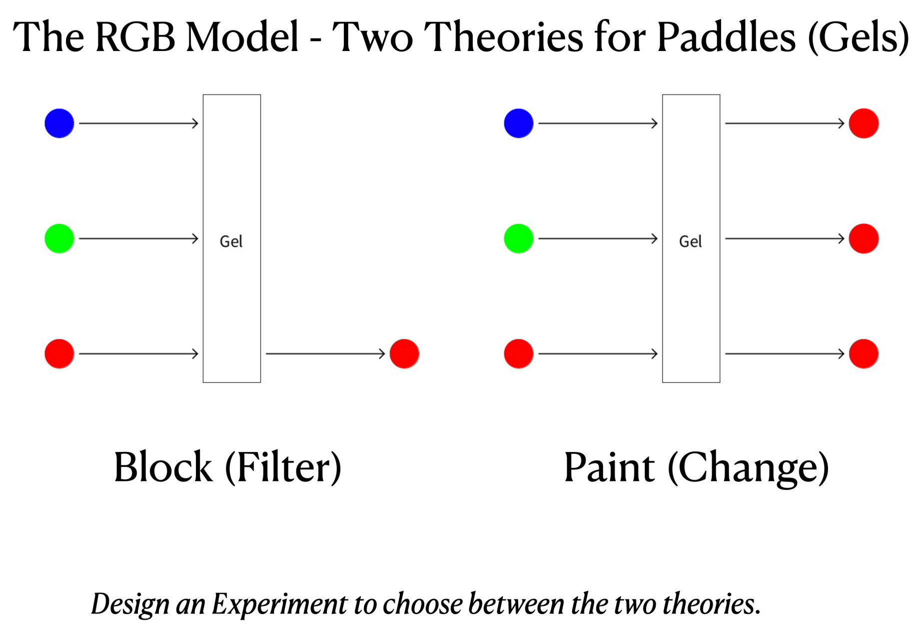
Slide 18

My own approach to understanding quantum mechanics begins with light and optics. I like the set of color filters, diffraction gratings and polarizing filters made by Rainbow Symphony and sold by them and by Amazon for under $20.00. The price has been rising. I bought them a few months ago for under $15.00.

https://www.rainbowsymphony.com/products/color-paddle-set

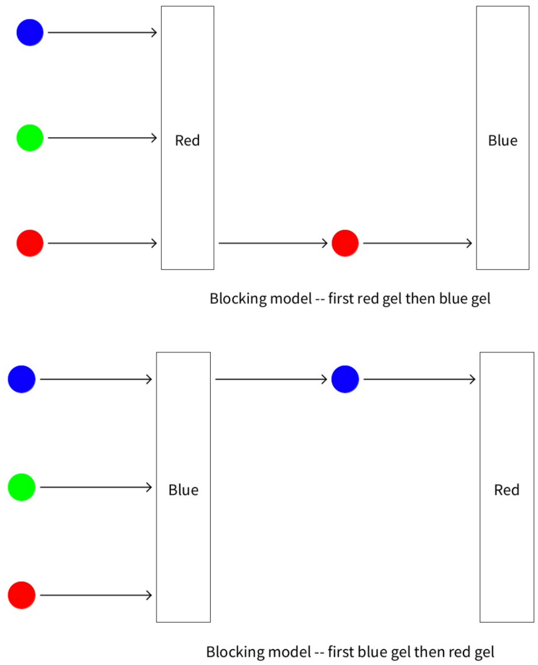
In class you can also use cheaper squares cut from larger sheets available from the same source. I like this source because they also include the transmission spectra for the color filters.

On the next slide we'll do a simple experiment using an easily observable feature of light, color, to practice the scientific method – that is, the dance between theory and experiment/observation.



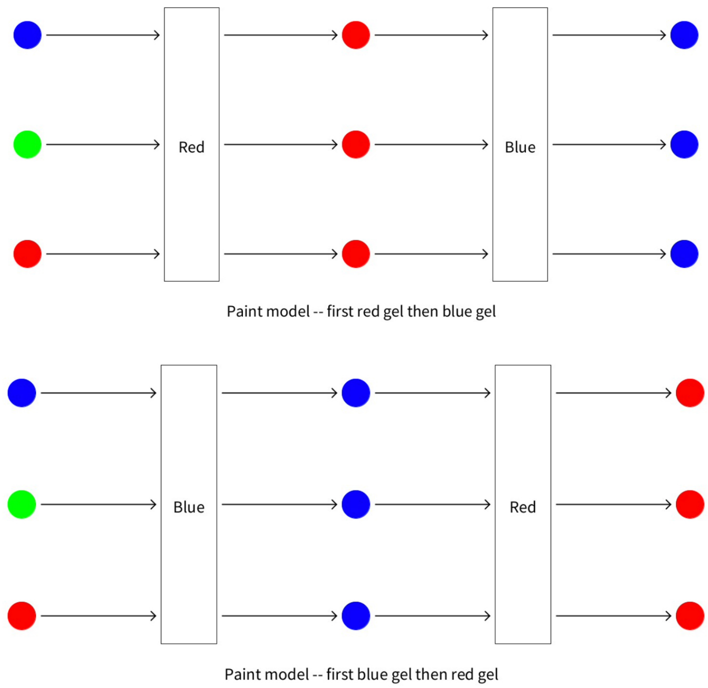
Slide 19

We begin with a simple model for color. There are three kinds of light – red, green and blue. These three colors are "seen" by us via the three different kinds of color sensors (red cones, green cones and blue cones) in our retinas. Our LED computer displays produce color via three different kinds of LEDs, one for each of red, green and blue light. Three of the paddles in the paddle set are red, green and blue gels. The figure above shows two models of how these gels work. The model on the left – the "filter" model – says that each gel blocks some colors and lets others through – for example, the red gel blocks blue and green light but lets red light through. The model on the right – the "paint" model – says that each gel changes some colors and lets others through – for example, the red gel lets red light through but paints green and blue light red. Take a few minutes to design an experiment that enables us to choose between these two models (theories).



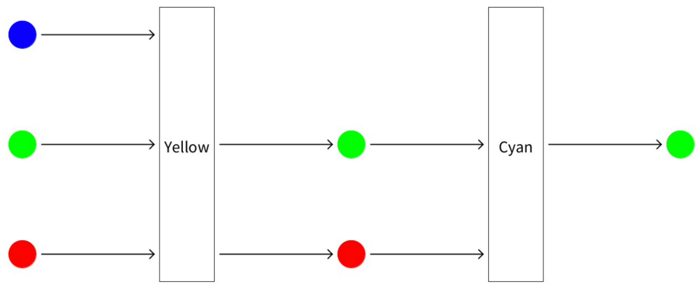
Slide 19a

In this two-part experiment we make a sandwich with one red and one blue gel. If white light passes through this sandwich then the blocking model predicts the output will be black and it doesn't matter whether the while light passes through the red gel first and the blue gel (top) or passes through the blue gel first and then the red gel (bottom).

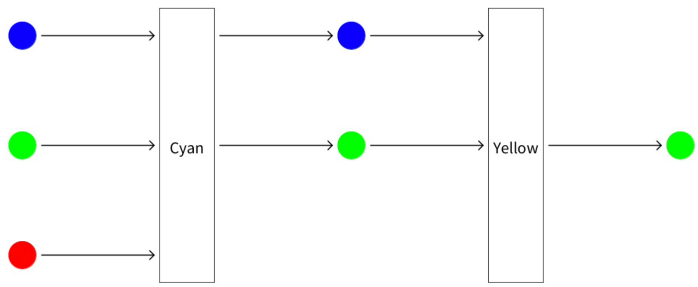


Slide 19b

But, the paint model implies that if white light passes through the red gel first and then the blue gel the output will be blue but if it passes through the blue gel first and then the red gel then the output will be red (bottom).



Blocking model -- first yellow gel then cyan gel



Blocking model -- first cyan gel then yellow gel

Slide 19c

The color yellow is a mixture of equal amounts of green and red. According to the blocking model the yellow gel blocks blue and allows red and green to pass through.

The color cyan is a mixture of equal amounts of blue and green. According to the blocking model the cyan gel blocks red and allows blue and green to pass through.

According to this model when white light passes through a yellow cyan sandwich the output is green and it doesn't make any difference whether it passes through the yellow gel first or the cyan gel first.

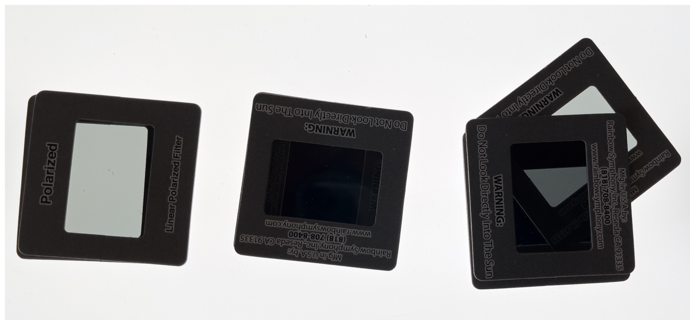
Slide 20

Here is one experiment – place a red filter on top of a blue filter and look through the the combination. The photograph above shows that the blocking model is better than the paint model.

Slide 21

The same model can explain this photograph.

Polarized Light

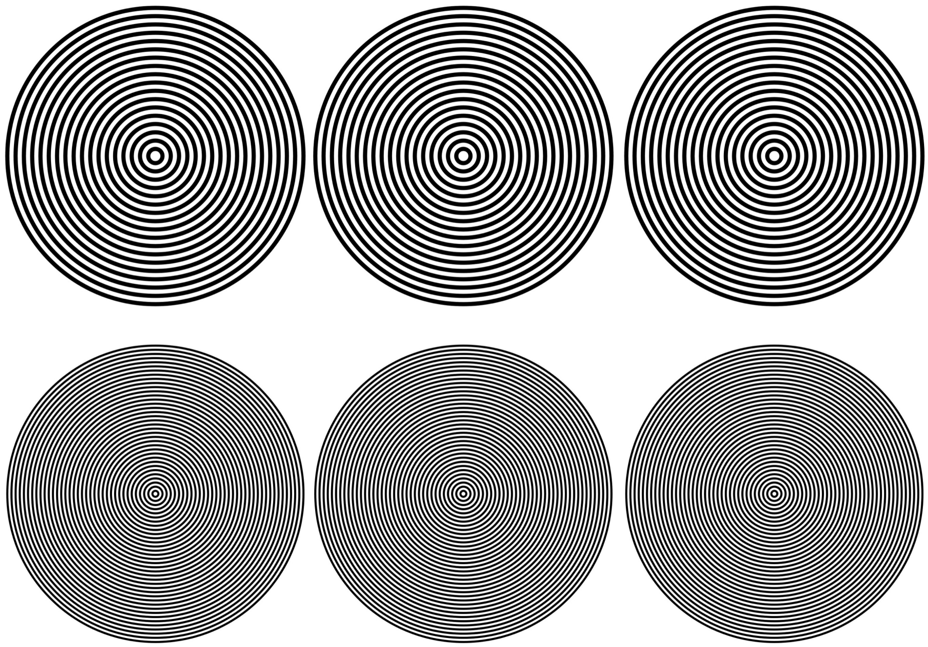


Slide 22

As a next step building towards an understanding of quantum phenomena lets look at polarized light. Polarization is less visible than color but most of us have experienced it through Polaroid$^{\text{TM}}$ sunglasses. The paddle set includes two polarizing filters or for my classes I like to use polarizing filters mounted in slide mounts, or just squares of polarizing filters. If you have one or two polarizing sunglasses handy you can do a surprising experiment. The easiest way to do this experiment uses your computer if it has an LED screen. Look through the Polaroid$^{\text{TM}}$ sunglasses at your computer screen and rotate the glasses as you look. You should notice that when the glasses are in the usual (horizontal) position you can see the screen but when they are rotated 90° they block the screen. The light from the LED screen is polarized in the horizontal direction.

Hold the second pair of Polaroid$^{\text{TM}}$ sunglasses at a 45° angle and look through them at the screen. Now try two experiments – one with the 45° angle Polaroid$^{\text{TM}}$ sunglasses between your eyes and the 90° angle Polaroid$^{\text{TM}}$ sunglasses – and the second with the 45° angle Polaroid$^{\text{TM}}$ sunglasses between the screen and the 90° angle Polaroid$^{\text{TM}}$ sunglasses.

We need a model that is more like the “paint” model than the “filter” model and one that can express direction as well as magnitude – perhaps using vectors or complex numbers.



Slide 23

The next set of experiments I usually do uses the flashlight apps on cellphones and your hands to project shadows on the wall and then laser pointers and diffraction gratings to “project” diffraction patterns on the wall. The paddle set has diffraction gratings or you can get cheap “diffraction glasses” from Rainbow Symphony or other sources like Amazon. I use laser pointers that are sold as pet toys for about \$12.00 for a a set of three.

Now we need the “wave” model to explain what we see. You can make a simple experiment exploring interference by printing two copies of the figure above, one on ordinary paper and the other on transparency film. Lay the transparency film on top of the plain paper and then slide it around.

We’re still some distance from our goal of understanding quantum phenomena but we’ve come a long way and set the stage.

Descriptive vs. Explanatory

Too Many Parameters -- Falsifiable?
(Lee Smolin, *The Trouble With Physics*)

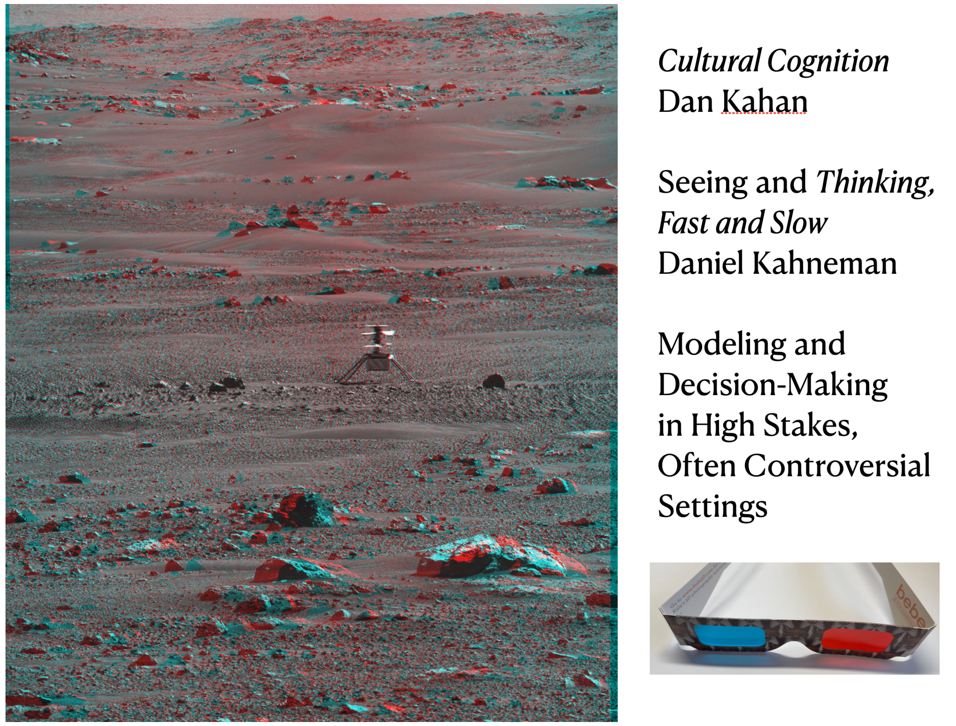
Machine Learning vs Human Learning

Slide 24

One of the most useful distinctions in modeling is between *descriptive* models like Kepler's model of planetary motion and *explanatory* models like Newton's model, which shows how planetary motion can be explained by the laws of motion and gravity. We need explanatory models both to build understanding and to make predictions in new settings.

Some models are based on a small number of principles – for example, Newton's laws of motion and gravity – that are used to make many different predictions. These laws have a small number of parameters – for example, mass. Other models have more principles or many parameters and can easily be adjusted to fit many different situations. This is not new – for example, one criticism, articulated by Lee Smolin, of String Theory is that it can so easily explain anything that it is essentially not "falsifiable" – that is, it cannot be tested by experimentation and observation.

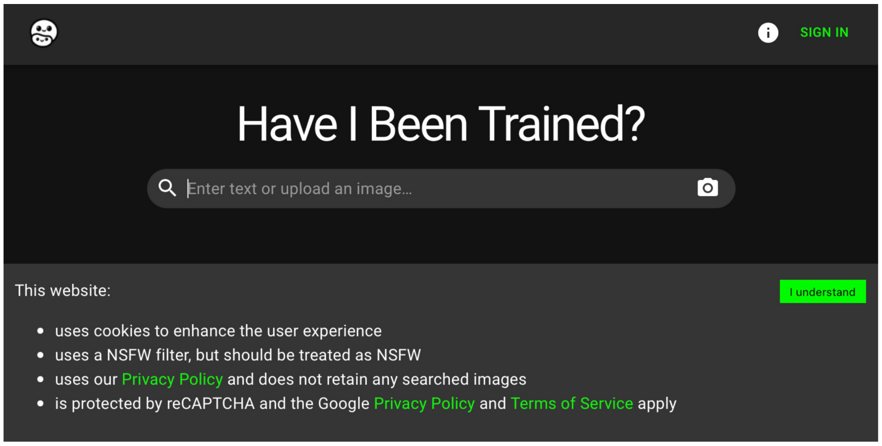
Machine learning suffers from both flaws – it is descriptive rather than explanatory and has so many parameters that it can predict anything. Nonetheless machine learning can be useful in the same way that Kepler's laws were useful as part of the scientific trip towards Newton's laws.



Slide 25

When most people are asked what parts of our body are used for seeing they answer "our eyes." If you look at the picture above through a pair of red-cyan 3D glasses you will see that your brain plays a large part. Your two eyes report two, two-dimensional images to your brain and your brain "sees" a three-dimensional model – in this case, the Ingenuity helicopter on Mars. Even the data that comes from direct observation – sight, sound, smell, taste and touch – is processed by our brains at a subconscious level. The *Cultural Cognition Project* at Yale University by Dan Kahan and others provides lots of insight into how culture affects perception and is particularly germane in this era of "alternate facts."

Daniel Kahneman's work and his book *Thinking Fast and Slow* can give us additional insight into decision-making and the roles of artificial intelligence/machine learning on the one hand and human intelligence on the other. "Fast thinking" refers to the wide-ranging, highly parallel background thinking that is always going on and "slow thinking" refers to the more focused, "rational," linear thinking of which we are more conscious. This dichotomy is often accompanied by a narrative in which the slower, more rational, thinking keeps the knee-jerk fast thinking in check. In fact, the two are synergistic. We need the highly parallel background thinking for its ability to survey a lot of disparate information and to spot connections. AI/ML with their "fast thinking" characteristics and the more rational linear thinking of humans may work together in the same way.



A New Element -- AI/ML driven actors that pass the Turing test (but so did George Santos)

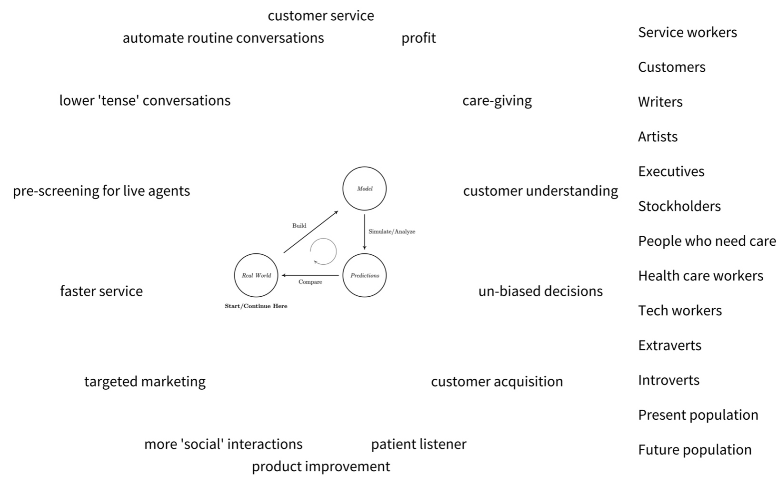
- ChatGPT
- Text-based image generation
- Text-based or image based image search -- https://haveibeentrained.com/

Slide 26

Now we return to the overarching theme of this discussion – using differential equations as one tool in a modeling toolkit to help us understand our world and make wise personal and public policy decisions. We do this in a new context – making wise decisions about how we use“robotic” agents that are based on artificial intelligence and machine learning – like ChatGPT, text-based image generation and text-based image search. You've almost certainly been deluged by information about the first two and played with at least the first one. The third one is also very interesting and is being used by visual artists whose work is probably being used illegally and is certainly being used unethically

The common theme is that these technologies pass the Turing test – they can lull people into thinking they are dealing with other people. This is as much a commentary on other people as on the technology – George Santos apparently passed the Turing test.

If you haven't been paying attention to these technologies it is really important to start now. They are rapidly changing as their creators fine-tune them and their flaws are more immediately visible now than they will be soon.



Modeling, Goals and Stakeholders (AI/ML)

Slide 27

One of the topics after the break will be using this framework to understand and make wise decisions about how we use “robotic” agents powered by AI/ML

Are We Bits or Qubits?

Slide 28

I want to close with the observation I made earlier – AI/ML and quantum technologies will give us new and potentially very powerful modeling paradigms and analogies.

Differential Equations: One Course in a Lifetime of Modeling

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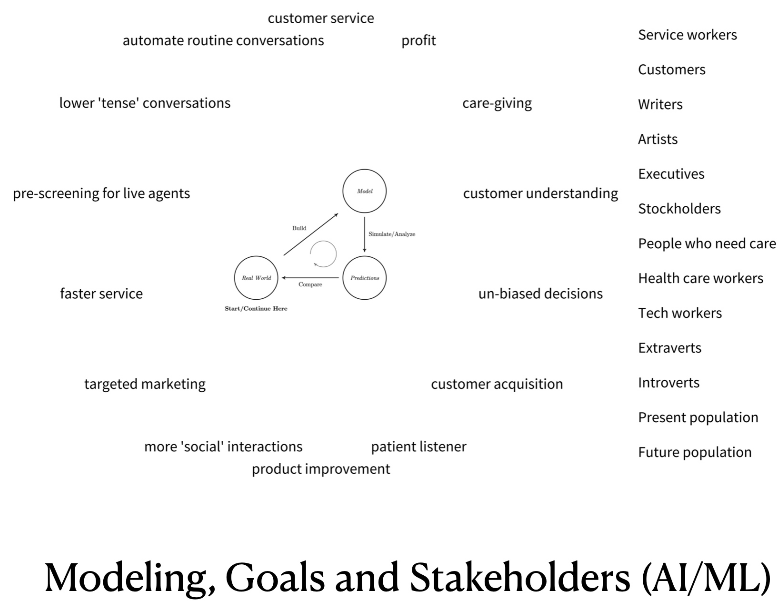
Modeling is essential for understanding and bettering our lives
EXPLANATORY vs DESCRIPTIVE -- GOALS and VALUES

Slide 29

Once more – my contact information.



Slide 30

It's time for a five minute break before we return for a discussion